Multilevel Methods Applied to the Design of Resonant Cavities

S. Costiner, F. Manolache, and S. Ta'asan

Abstract—**An application of multilevel (ML) methods to compute the modes and eigenvalues of resonant cavities is presented. The involved methods include an ML eigenvalue solver, an ML mode separation technique, a boundary treatment method, and a subspace continuation technique (SCT) for sequences of problems. In the presented numerical experiments, an asymptotic convergence factor of order 0.1 is obtained for ML cycles on all fine levels, while performing only a few relaxations per cycle. This factor is obtained for a rectangular cavity as well as for cavities having reentrant corners, holes and narrow regions, and presenting clusters of close and equal eigenvalues. A second order scheme is obtained for the computed eigenvalues and modes with an amount of work of order $O(qN)$ for q modes of size N on the finest level. The SCT is illustrated on a moving boundary problem, where solutions change fast at a small boundary change. Such computations are applied to the design of new microwave selective devices.**

Index Terms—**Maxwell equations, microwave device, resonant cavity, multilevel numerical techniques, multigrid, eigenvalue algorithms, continuation techniques.**

I. Introduction

THE DESIGN of resonant cavities is important in many applications such as filters, multiplexors, etc. A usually complicated and time consuming design step is the numerical approximation of the cavities' modes and eigenvalues. A main numerical approach is to represent the problem on a single-level, e.g., by finite differences or finite element discretizations, and to solve it through some algebraic procedures [1]. The discretized problems should have large sizes in order to obtain accurate modes and eigenvalues. Typically a large amount of work is required to solve these problems, mainly due to: 1) the slow convergence rate of single-level iterative methods; 2) the sequence of increasingly larger problems to be solved in order to obtain the desired approximation of continuous solutions from the sequence of discrete solutions; 3) the necessity to solve sequences of eigenvalue problems, e.g., for different domains or parameters required in design.

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In multilevel (ML) techniques the continuous problem is represented on several coarser and finer levels, and the solution process uses interactions between the levels. ML techniques generally show a faster convergence rate than single-level techniques, provide convergence towards the continuous solution directly from the solutions on the different levels [2], and allow a more efficient treatment of problem sequences. This is mainly due to the reduction of the expensive fine level work to cheap coarse level work, e.g. in approximating the smooth solution components on coarse levels [3].

In this paper we apply an ML eigenvalue technique coupled with a boundary treatment method and a subspace continuation technique (SCT), presented in Section II, to the design of resonant cavities. The boundary treatment method is used to achieve good convergence rates on domains with reentrant corners, holes and narrow regions, and where the representations of the cavity and the solutions differ strongly on different levels. The SCT is used in solving sequences of problems. Specially difficult cases involving close and equal eigenvalues are discussed in Section IV. The presentation is selfcontained and aimed towards the worker in the field, more theoretical aspects being presented in the reports [4], [5].

The ML techniques are used for the design of microwave selective devices proposed in [6]. These devices, briefly presented in Section III, are based on weakly asymmetric resonant cavities (WARC) which admit close frequency modes with localized amplitudes. Short conclusions are presented in Section V.

II. Algorithm Description

This section describes a general ML algorithm to compute several required modes and eigenvalues of a discretized eigenvalue problem, and discusses its efficiency.

For generality, the eigenvalue problem to be solved is

$$Lu - \lambda u = 0, \tag{1}$$

where L is the operator whose modes u and corresponding eigenvalues λ are sought. The ML algorithm is formulated for general matrices L, and in our applications L is a discretization of a partial differential eigenvalue problem derived from the Maxwell's equations for the electromagnetic field in a resonant cavity. Generalized eigenvalue problems of the form $Lu = \lambda Bu$ can be analogously treated. Assume that the required information is the q eigenvalues λ_i of L with largest absolute values, and their corresponding modes $u_i, i = 1, \cdots, q$. To compute other eigenvalues, L can be replaced with an operator

with the same modes, e.g., $I-\omega(\mu I-L)$, where ω and μ are rescaling and shifting constants.

To describe the ML algorithm a single-level algorithm [7], often called subspace iteration or Ritz iteration, is first considered. Denote by $U^n=(u_1^n,\cdots,u_q^n)$ the matrix whose columns u_i^n are the approximations of the required modes at the computational step n. The subspace iteration algorithm starts with a random U^0 and iterates the following three steps: (i) $U^n=LU^{n-1}$; (ii) orthonormalize the columns of U^n; (iii) separate the solutions in $U^{n+1}=U^nE$ by a Rayleigh Ritz (RR) type projection. U^n is an approximate basis of the space generated by the first q modes. The modes are approximated by the linear combinations U^nE where E is a $q\times q$ invertible matrix to be found. The RR projection for U^n usually finds E and a diagonal matrix $\Lambda=\mathrm{diag}(\lambda_1,\cdots,\lambda_q)$ requiring that U^nE are approximate modes associated with Λ

$$LU^nE=U^nE\Lambda, \tag{2}$$

and solving the small $q\times q$ generalized eigenvalue problem

$$(U^n)^T(LU^nE-U^nE\Lambda)=0. \tag{3}$$

The convergence of the subspace iteration is fast if the first q eigenvalues have large absolute values compared to the others, due to step (i). If λ' approximates well the eigenvalue λ, then the matrix $L'=(L-\lambda' I)^{-1}$ has an eigenvalue $\mu=1/(\lambda-\lambda')\approx\infty$ and also has the same modes as L. Thus, if $q=1$, the above algorithm using L' instead of L will converge fast to the accurate mode associated with λ. In this case, the step (i) for L' replacing L becomes

$$U^n=(L-\lambda' I)^{-1}U^{n-1}. \tag{4}$$

The iterations (4) converge in only a few steps but often raise huge computational difficulties when λ' approaches an exact eigenvalue since the matrix $L-\lambda' I$ is almost singular, and when L has large size as needed in the cases we consider. One approach to overcome these difficulties is to write (4) as

$$(L-\lambda' I)U^n=U^{n-1}, \tag{5}$$

and to solve it iteratively. The efficiency of solving (5) iteratively by ML algorithms has been often shown [2], [3], [8].

A new ML algorithm, also suitable for close and equal eigenvalues as often appear in microwave devices, is presented next. In the ML algorithm the problem (1) is discretized on a sequence of increasingly finer grids. The spaces of discretized functions corresponding to the different grids will be called levels, level 1 being the coarsest one. Finite differences, finite elements, or other discretization types can be used to define the levels. First, the problem is solved on level 1, here the work being cheap since the level is coarse. The solution is then interpolated to level 2 where it is corrected by a few ML cycles involving level 1, then the solution is interpolated to the finer level 3 where the solution is corrected by ML cycles involving the coarser levels, and the algorithm continues in the same way until the finest level. An ML cycle involving only two levels is presented next. Denote by L_1, L_2 the discrete operators, by U_1, U_2 two approximate solutions for the two levels, and by I_j^k a transfer operator from level j to level k (e.g., interpolation). The cycle starts by relaxing U_2 several times. Then the level 2 problem

$$L_2U_2-U_2\Lambda=T_2=0 \tag{6}$$

is transferred to level 1 as

$$L_1U_1-U_1\Lambda=T_1, \tag{7}$$

where

$$T_1=L_1I_2^1U_2+I_2^1(T_2-L_2U_2). \tag{8}$$

On level 1 the solutions are separated by $U_1\leftarrow U_1E$ where E is obtained as above by an RR type projection of the form

$$U_1^T(L_1U_1E-U_1E\Lambda-T_1E)=0. \tag{9}$$

This is a generalized RR type projection referred to subsequently as GRR [4]. Then U_1 is improved by several relaxations and is used to correct U_2 according to

$$U_2=U_2+I_1^2(U_1-I_2^1U_2). \tag{10}$$

The cycle ends by relaxing again U_2 several times. Due to the introduced T terms, the problems (e.g., 6, 7) have the same form on all levels. This allows to define an ML cycle as above: on each level the solutions are relaxed, transfered to coarser levels, corrected by coarser level solutions and, on some intermediate levels, separated by a GRR.

Usual difficulties encountered by eigenvalue solvers, and new techniques used to overcome them are discussed next.

The mixing of solutions corresponding to close or equal eigenvalues slows or even prevents the convergence. To treat this, we approximate together with the required modes all the other modes which are mixed with them during processing. The solutions are treated simultaneously during the cycles, usually using the projections to separate them on coarse levels. In this way the algorithm recovers the efficiency obtainable for well separated eigenvalues. These cases are relevant for resonant cavities.

The cavity geometry features and the differences in the cavity representations on different levels are obstacles for convergence and efficiency. We treat these difficulties by relaxing the solutions on the layers along the boundaries after interlevel transfers. Several relaxations are performed on a few layers along the boundaries, each layer in turn, starting with the layer closest to the boundary. This procedure we call *boundary layer relaxation* (BLR) [5]. In the numerical experiments to be presented it is shown that using BLR, the full efficiency obtained in the regular cases (rectangular domains) can be recovered for domains with reentrant corners, holes, narrow regions and whose representations differ considerably on different levels.

The solution's accuracy is provided by the ML cycles which improve the approximations of subspaces, the ML projection which separates the modes, and the BLR, which damps the large errors along the boundaries. The set of solutions on consecutive levels can be used to check the convergence of the discrete solutions towards the continuous ones that can be

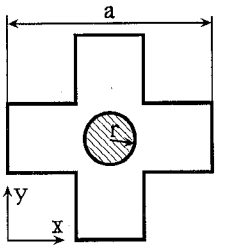

Fig. 1. Section of a cross-shaped closed cavity (CSC) having identical arms, low height, and a conductor rod whose displacement is used to tune the cavity.

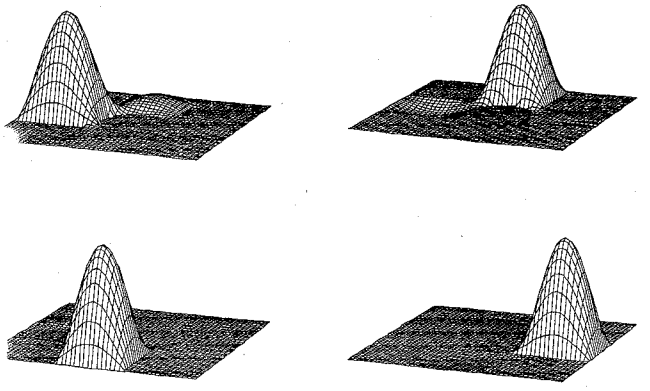

Fig. 2. The first four modes of a CSC having the rod's center slightly asymmetrically placed at (0.48, 0.49) a. The plots represent the electric field distribution in the modes, the magnetic flux lines consisting of level lines of these distributions.

TABLE I
THE FIRST TWO CLUSTERS OF FOUR APPROXIMATED EIGENVALUES OF THE CSC WITH $R = 016a$ AND CENTER (0.48, 0.49)a

No.	eigenvalue
1	-0.15860993687992E+04
2	-0.16438848116572E+04
3	-0.17370340398353E+04
4	-0.17694738183819E+04
5	-0.36184697368680E+04
6	-0.38457723855499E+04
7	-0.42536026937718E+04
8	-0.43475194074468E+04

approximated by extrapolation techniques (e.g., of Richardson type).

This algorithm can be naturally coupled with *subspace continuation techniques* (SCT) for the efficient solving of problem sequences required for the design process (e.g., parameter tuning or optimization). The SCT use the space of previous solutions to approximate the new solutions and correct the new solutions by ML cycles. For example, suppose that approximate solutions (modes and eigenvalues) were computed for a parameter value. The SCT computes the solutions for a new parameter value by: start with the previous solutions, update the parameters, relax the solutions several times, correct the new solutions by several ML cycles. Several parameters can be updated at a time (e.g., a boundary part). The solving efficiency for problem sequences is determined by performing cheap SCT instead of solving a new problem each time, and by solving most of the problems in the sequence

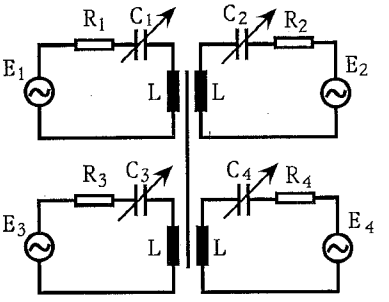

Fig. 3. The equivalent circuit for CSC. The rod position is modeled by the simultaneous tuning of the four capacitors.

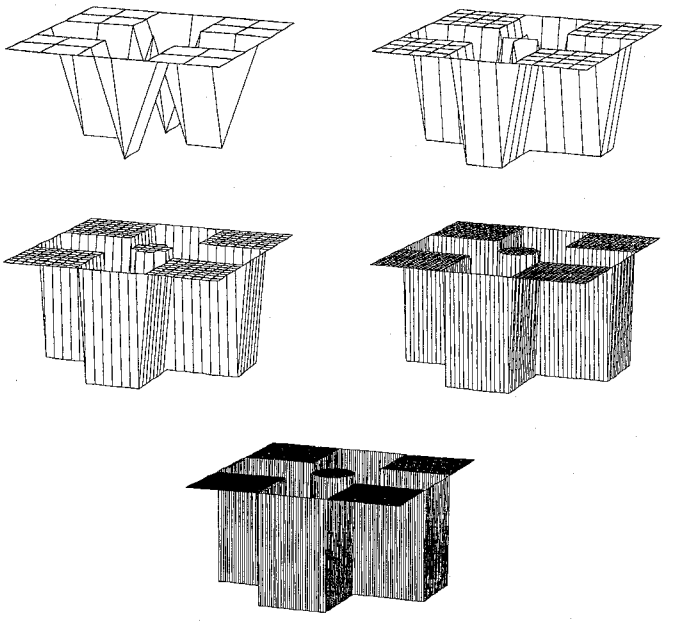

Fig. 4. The five representations of the discretized closed CSC1, on 5 levels.

only on coarse levels, (e.g., for intermediate problems or for parameters which are not of final interest).

This algorithm differs from previous algorithms mainly by BLR and by SCT. The algorithm is presented in a general algebraic form and can be used in the same way to compute real or complex modes and eigenvalues for a larger class of problems from different engineering domains.

An estimate of memory required to compute q modes of size N on the finest level is of order $O(3qN)$. An estimate of work to solve one eigenvalue problem is $O(N)$ operations per mode. If a finest level high accuracy separation is needed then the work may be of order $O(q^2N)$ for each complete cluster (set of close or equal eigenvalues) of q modes. The presented ML techniques may be several orders of magnitude more efficient (faster) than the single-level techniques (as usual for ML techniques, see the numerical examples). This is mainly due to 1) the smooth components of the solutions are efficiently approximated on coarse levels (these components converge usually very slow in single-level techniques), 2) the usage of ML cycles which approximate the fast convergent inverse power iterations, 3) solving sequences of problems mostly on coarse levels by SCT, and 4) the convergence and the accurate approximation of the continuous solutions are obtained from the sequence of solutions computed on different levels.

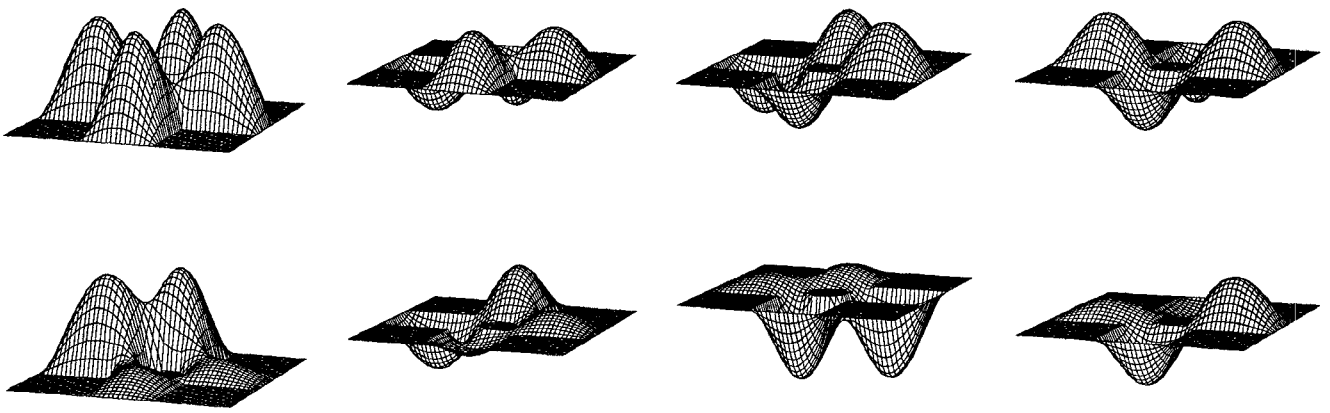

Fig. 5. The first four modes of the CSC1 (upper row), and of the CSC2 (lower row) on level 4.

III. WARC Design and Applications

WARCs are branch shaped closed cavities, the branches being weakly coupled and having close resonant frequencies [6]. An example of a WARC is the cross-shaped cavity (CSC) (Fig. 1). The CSC has four weakly coupled branches, the rod position being used for tuning.

The useful property of some WARC's consists in the existence of clusters of localized modes with close eigenvalues. For example, the first four modes of a weakly asymmetric CSC have the electric field distribution shown in Fig. 2, and the corresponding eigenvalues belonging to one cluster (Table I). The modes localization makes WARC's suitable for efficient multiple output band-pass filters, multiplexors, and switches, which have the splitting, coupling and resonant structures integrated in the same cavity. For example, a CSC presenting an input gate in one of the branches and output gates in each of the other branches can be used as a multiple output filter [6]. The tuning by the rod's positions can be simulated using the SCT technique described in Section II and illustrated in Section IV-B, (e.g., the variation of the eigenvalues and of the couplings between modes can be analysed using the SCT).

The design of such a device is done using an equivalent circuit [9]. The numerical values of the circuit's components are derived from the cavity's eigenvalues and modes of interest. Design difficulties result from the fact that the equivalent circuit of a WARC is not an RLC group, as usual, but many such groups weakly coupled to one another [10]. Further the eigenvalue algorithms encounter the difficulties cited in Section II, and additionally, for small cavity asymmetry, the localized modes are strongly sensitive to small shape perturbations, as illustrated in Section IV.

An equivalent circuit for the CSC is shown in Fig. 3, where the cavity's feeding gates are replaced for simplicity by signal sources. The four RLC groups are magnetically coupled. The way of coupling reflects the magnetic flux line geometry in the cavity (Fig. 2).

The numerical analysis of a WARC consists mainly of the computation of the eigenvalues and modes for the symmetric and weakly asymmetric cavities. The results of such computations for several non-trivial cases, using the ML algorithm described in Section II, are presented in the next section.

IV. Computational Examples

The efficiency of the presented algorithm is illustrated by a series of numerical experiments described in the following. For a symmetric CSC, it is shown that the usual multigrid convergence rate, accuracy, and minimal work [2] are obtained by our algorithm, for cases presenting boundary difficulties, eigenvalue clustering, and coarse level representation difficulties. For an asymmetric CSC, the efficiency of the SCT is shown. Finally, for a rectangular cavity, a second order scheme in approximating $TE_{m,0,p}$ modes is obtained, and it is shown that the same computational efficiency is achieved for this regular case as well as for the CSC.

For all the presented cases, L from (1) is a finite difference discretization of the Laplacian on rectangular grids. The four eigenvalues of L having the smallest absolute value and their corresponding modes are computed. The ML cycles included two relaxations per fine level (levels 3, 4, 5), i.e., one relaxation on the path down, from fine to coarse, and one relaxation on the path up, from coarse to fine. Such cycles are called $V(1,1)$ (indicating the number of relaxations). On the coarse levels 1 and 2, several relaxations were done. The GRR projection was performed only on the coarse levels.

The problem was discretized by finite differences on rectangular grids. The number of points on coarsest grid, level 1, was 7×7 ($n = 7$ nodes in each direction) thus the mesh size was $h = a/6$. On each finer grid the mesh size was two times smaller, thus the number of grid points in each direction on level $m = 2, 3, \cdots$, was $2^{m-1}(n-1)+1$, i.e., 13, 25, 49, 95, 189 points. The modes were equal to 0 on the boundary. The Laplacian was discretized by the usual five point scheme, $(\Delta_h U)_{ij} = (U_{i,j-1} + U_{i,j+1} + U_{i-1,j} + U_{i+1,j} - 4U_{ij})/h^2$, where h denotes the grid's mesh size. The transfer operator from a level to the next finer one was the linear interpolation, during the ML cycles. The transfer operator from a level with mesh h to a coarser one with mesh $2h$ was the transposed of the linear interpolation, i.e., $U_{ij}^{2h} = (4U_{ij}^h + 2U_{i,j-1}^h + 2U_{i,j+1}^h + 2U_{i-1,j}^h + 2U_{i+1,j}^h + U_{i-1,j-1}^h + U_{i-1,j+1}^h + U_{i+1,j-1}^h + U_{i+1,j+1}^h)/16$. The relaxation on coarsest level at the start of the ML algorithm was the shifted power iteration, during the ML cycles the Gauss-Seidel relaxation (in red-black ordering)

TABLE II
THE RESIDUALS AT THE BEGINNING AND END OF 13 CYCLES, ON 5 LEVELS, FOR THE FIRST 4 MODES OF CSC1. THIRTEEN CYCLES REDUCE THE RESIDUALS TO ORDER 10^{-10} ON ALL LEVELS

mode	level 1	level 2	level 3	level 4	level 5
1	.10E+3 .23E-13	.18E+3 .76E-12	.25E+3 .32E-11	.32E+3 .16E-10	40E+3 10E-09
2	.12E+3 .84E-14	.18E+3 .14E-11	.21E+3 .47E-11	.28E+3 .14E-10	.35E+3 .89E-10
3	.91E+2 .89E-14	.19E+3 .77E-12	.21E+3 .30E-11	.28E+3 .72E-11	35E+3 .58E-10
4	.11E+3 .86E-13	.20E+3 .11E-11	.16E+3 .33E-11	.22E+3 .12E-10	.29E+3 .54E-10

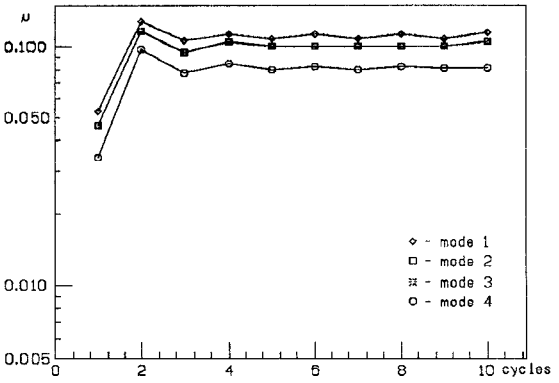

Fig. 6. The asymptotically constant convergence factors μ of $V(1,1)$ ML cycles, close to 0.1, for the first 4 modes of the CSC1, on level 5.

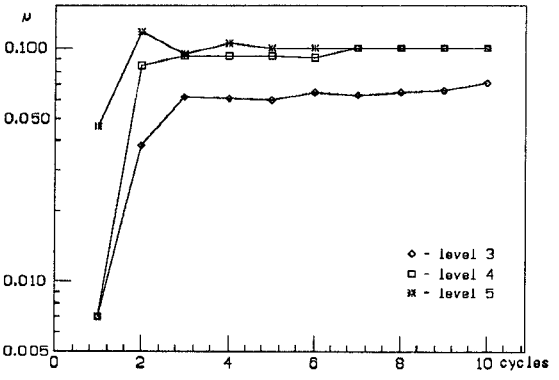

Fig. 7. The asymptotically constant convergence factors μ of $V(1,1)$ ML cycles, close to 0.1, on levels 3, 4, 5, for mode 2 of the CSC1.

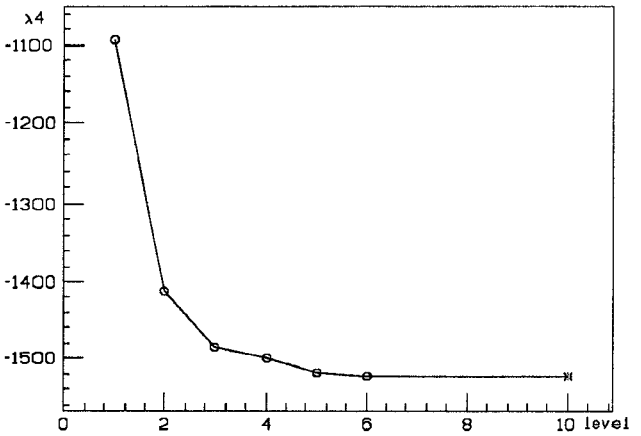

Fig. 8. The convergence of the eigenvalue 4 computed on 6 levels, and the approximation by extrapolation of the eigenvalue 4 of the continuous problem for the CSC1.

TABLE III
THE FIRST FOUR EIGENVALUES COMPUTED ON 6 LEVELS AND THE APPROXIMATIONS OF THE CONTINUOUS PROBLEM EIGENVALUES (c) FOR CSC1. OBSERVE THE EQUAL EIGENVALUES 2 AND 3 (WITH 12 COMMON DIGITS) ON ALL LEVELS AND THE FOUR DEGENERATE EIGENVALUES ON LEVEL 1

level	ev 1	ev 2	ev 3	ev 4
1	-1094.26878334	-1094.26878334	-1094.26878334	-1094.26878334
2	-1324.94636653	-1368.07131712	-1368.07131712	-1411.97505974
3	-1381.61635536	-1433.05778787	-1433.05778787	-1483.96253809
4	-1373.23765043	-1436.27359781	-1436.27359781	-1498.87806401
5	-1410.06598639	-1463.59424179	-1463.59424179	-1517 31161962
6	-1417.10678892	-1469.40380631	-1469.40380631	-1521.99808316
c	-1419.4537	-1471.3403	-1471.3403	-1523.5602

TABLE IV
THE SCALAR PRODUCTS OF CSC1 MODES REPRESENTED ON LEVEL 5 AFTER 10 CYCLES. THE RR TYPE PROJECTION WAS PERFORMED ON LEVELS 1 AND 2. IT IS REMARKABLE THAT THE DEGENERATE MODES 2 AND 3 ARE ACCURATELY ORTHOGONAL. USUALLY A FINE LEVEL ORTHOGONALIZATION WITHIN THE DEGENERATE SUBSPACE IS REQUIRED TO OBTAIN ORTHOGONALITY

mode 1	mode 2	scalar product
1	2	0.11E-13
1	3	0.24E-13
1	4	0.43E-15
2	3	-0.11E-13
2	4	0.38E-14
3	4	-0.35E-14

was used [7]. The RR projection was performed using a standard package of algebraic subroutines.

A. Symmetric CSC

A symmetric CSC of size a denoted CSC1, having a central rod of radius $r = 0.1a$ is considered. (In all presented cases, a is arbitrarily chosen $a = 2\pi/10$. The a can be used to rescale the wavelength.) The discretizations of CSC1 on 5 levels are shown in Fig. 4.

The modes computed on level 4 are shown in the upper row of Fig. 5. An asymptotic convergence rate, close to 0.1, is obtained on the fine levels (Figs. 6 and 7). A few cycles were sufficient to obtain accurate modes and eigenvalues on all levels (Table II). Fewer cycles (1–5) are sufficient to provide accurate eigenvalues and good approximations of modes. The convergence rate of the first cycle on each fine level was of order 0.01 (Fig. 7).

The continuous problem eigenvalues are approximated by extrapolation using the eigenvalues computed on 6 levels (Fig. 8, Table III). The approximated eigenvalues 2 and 3 resulted equal (12 common digits) on all levels (Table III). The mode separation by projections was performed on the coarsest levels only, and the resulting modes were accurately orthogonal even in the degenerate spaces (Table IV).

A comparison of convergence rates of our ML algorithm and of a single-level improved subspace iteration algorithm is shown in Fig. 9. The convergence rate of the single-level

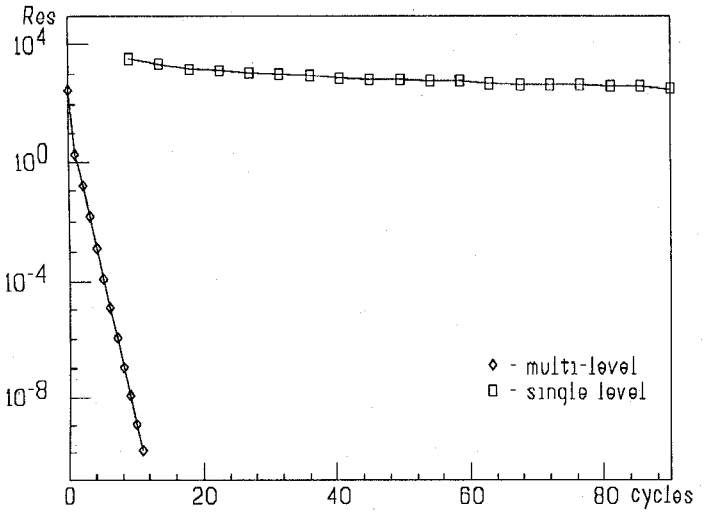

Fig. 9. The residuals of the ML algorithm and of the single-level improved algorithm for the mode 2 of CSC1 on level 4. The ML algorithm results more than 100 times faster than the single-level algorithm.

algorithm was 0.98 on level 4, thus 140 single-level cycles provided the residual decrease 0.1 of a single ML cycle. This situation becomes worse on finer levels since the convergence rate of the single-level algorithm tends to 1 as the level becomes finer, while the ML convergence rate remains of order 0.1. Moreover, a single-level step is usually more expensive than an ML cycle because the work involved in a single-level step is of order $O(q^2N)$ for q modes of size N, due to the orthogonalization and projection, while for an ML cycle it is of order $O(qN)$ if the projection is performed on coarse levels. Moreover, the ML algorithm begins with a lower residual than the single-level algorithm due to the initial approximation provided by the coarser levels. Many (hundreds on level 4) single-level iterations are needed to get the initial accuracy obtained by the ML algorithm.

B. The SCT for an Asymmetric CSC

The use of SCT is illustrated by computing the modes of an asymmetric CSC (denoted CSC2) starting from the set of solutions of CSC1. The CSC2 has the rod center at (0.48, 0.48)a, and its discretizations on four levels are shown in Fig. 10. The CSC2 modes on level 4 obtained after one SCT ML cycle are shown in the lower row of Fig. 5, the converged modes looking identically. The computed CSC2 eigenvalues after 1 or 2 cycles are very close to the converged eigenvalues obtained in several cycles only.

The SCT step consists in: 1) several BLR applied to the approximated CSC1 modes, providing an initial approximation of the subspace containing the CSC2 modes; 2) an ML cycle which separates the CSC2 solutions on coarse levels, and corrects them. Mode 1 of CSC1 on the four levels, and the approximated mode 1 of CSC2 in the path from coarse to fine level of the first cycle are presented in Fig. 11. Already on the coarsest level, the smooth component of mode 1 was approximated by the projection. In this case the coarse level projection is essential and more efficient than a fine level projection.

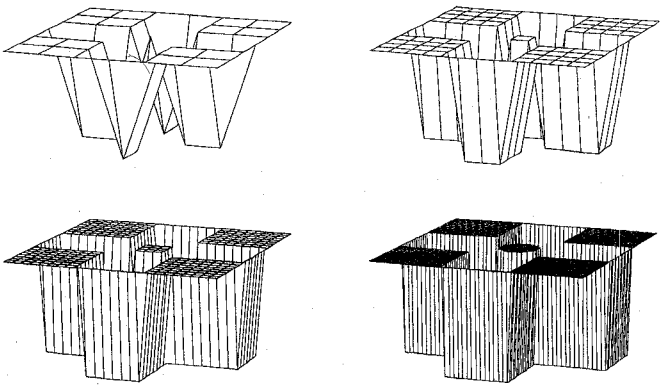

Fig. 10. The four representations of the discretized closed CSC2, on 4 levels.

BLR solved the difficulties resulting from the significant changes in solutions along the boundaries due to the differences in the rod discretization for CSC1 and CSC2. The large differences in the CSC1 and CSC2 solutions determined by the small change in the rod center on level 4 shows the robustness of the SCT in treating such cases as often appear in bifurcation regions.

C. $TE_{m,0,p}$ Modes in a Rectangular Cavity

The analytic discrete and continuous $TE_{m,0,p}$ modes and eigenvalues of rectangular cavities are known. This case also presents numerical difficulties because: 1) the eigenvalues $\lambda_{1,0,2}$ and $\lambda_{2,0,1}$ are equal; 2) mixing of modes occurs; 3) the eigenvalues on various levels are different; 4) the interpolation of an accurate solution on one level is not readily a solution on the next level. The solutions were approximated on 5 levels. The results showed that a second order scheme was obtained for modes and eigenvalues [5]. Already after the first cycle, on each level, the discrete eigenvalues were approximated by more than 10 digits and the maximum error between the computed and analytic modes was of order 10^{-4}. The resulting solutions were accurately orthogonal at the end of the first cycle on level 5. Using the sequence of eigenvalues computed on the five levels, the eigenvalues of the continuous problem can be approximated by extrapolation with a relative error of order 10^{-7}. Performing a larger number of cycles, a convergence rate of order 0.1 per cycle is observed on fine levels. This convergence rate, the behaviour of the results, and the amount of work required are close to those in the more difficult presented CSC1 case.

The efficiency of BLR is shown in Fig. 12. Often the ML convergence rate downgrades or the algorithms fail to converge if BLR are not used.

D. Observations on the Algorithm

The algorithm's efficiency can be further improved by several techniques such as: 1) Rayleigh type Quotients which approximate the eigenvalues of $Lu = \lambda u + T$ by $\lambda \approx u^T(Lu - T)/u^T u$; 2) the normalization of modes on different levels; 3) tuning of relaxation parameters; 4) additional modes introduced in computation to improve the convergence rate of the sought modes; 5) corrections with rotation, using the more

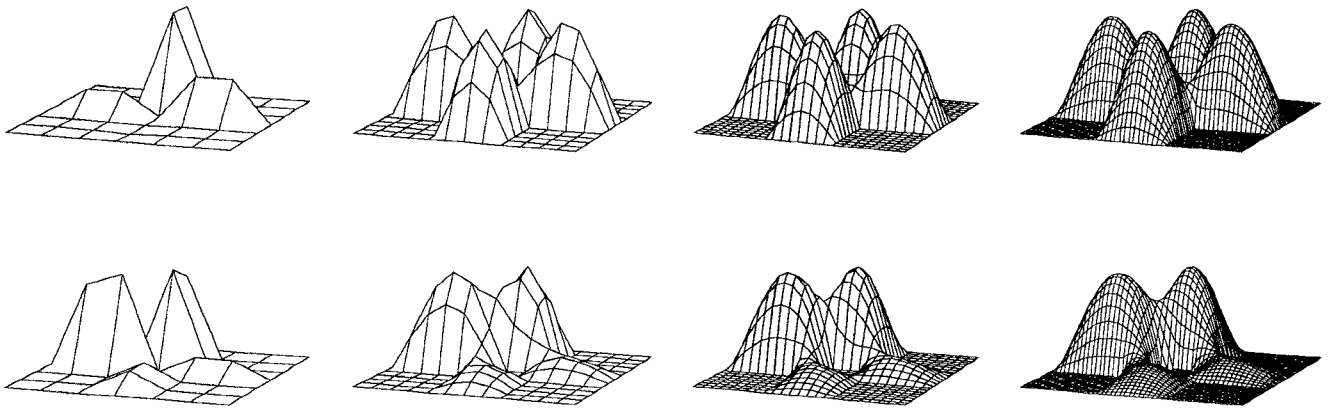

Fig. 11. Mode 1 on four levels, for the CSC1 (upper row), and for the CSC2 in the path from coarse to fine level of the first SCT ML cycle (lower row).

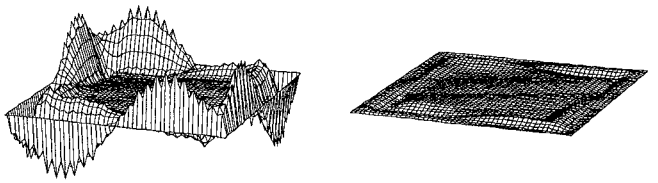

Fig. 12. The error of the approximated $TE_{2,0,2}$ mode of a rectangular cavity after 1 cycle without BLR (left), and after 1 cycle with BLR (right).

expensive formula $U_2 = U_2E + I_1^2(U_1 - I_2^1U_2E)$ instead of (10).

If one V cycle provides a 0.1 convergence rate then one *combined cycle* C_kV consisting of a combination of k cycles with different relaxation parameters can provide a convergence better than 10^{-k}, which one would obtain by applying k cycles V with fixed relaxations.

The algorithm can be generalized to solve nonlinear eigenvalue problems [11].

The numerical results presented in this section were obtained by the software [12]. The algorithm was developed to be used in the design of new selective microwave devices based on WARC [6], some of which are the subject of a patent application.

V. Conclusions

A multilevel method to compute the modes and eigenvalues of resonant cavities is presented. The problem is represented on a sequence of coarser to fine levels and the expensive fine level computations are much reduced to cheap coarser level computations. The algorithm can treat specific difficulties related to resonant cavities, such as close or equal eigenvalues, or difficulties induced by cavity shapes.

Sequences of problems resulting from a change of the cavity shape can be treated efficiently using SCT coupled with BLR, even in cases when small boundary changes imply large solution changes, as may happen close to bifurcation regions.

The numerical experiments described illustrate the method's robustness when it is used to obtain accurate solutions for very close or equal frequencies, and domains including reentrant corners, holes and narrow regions. In these experiments, an asymptotic convergence factor of order 0.1 is obtained for ML cycles on all fine levels, performing only a few relaxations per cycle. A second-order scheme is obtained for the computed eigenvalues and modes with an amount of work of order $O(qN)$ for q modes of size N on the finest level.

The algorithm has a general algebraic form. It can be used for various operators and discretizations (finite differences, finite elements, etc.) in other engineering applications.

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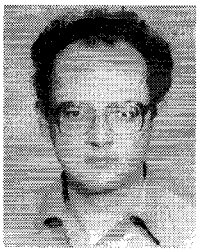

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